test 3 potential theory

2025

Testing time - 6 hours

Answer at least two questions.

The Dirichlet problem for the test

Let $\Omega \subset \mathbb{C}$ be a simply connected domain with a smooth boundary $\partial\Omega \in C^2$. Consider the Dirichlet problem:

$$\begin{aligned} &\Delta u(z) = 0, \quad z \in \Omega, \\ &u(z) = f(z), \quad z \in \partial\Omega, \end{aligned}$$

where $f \in C(\partial\Omega)$.

(1) Conformal Mappings and the Dirichlet Problem

(a) Let $\phi : \mathbb{D} \to \Omega$ be a conformal map, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that under the transformation $w = \phi(z)$, the Dirichlet problem on Ω is equivalent to solving:

$$\begin{aligned} \Delta v(\zeta) &= 0, \quad \zeta \in \mathbb{D}, \\ v(\zeta) &= g(\zeta), \quad \zeta \in \partial\mathbb{D}, \end{aligned}$$

where $g(e^{i\theta}) = f(\phi(e^{i\theta}))$.

(b) Derive the explicit relationship between $u(z)$ and $v(\zeta)$.

(2) Green's Function Representation

Let $G(z, w)$ be the Green's function for Ω, satisfying:

$$\begin{aligned} &\Delta_z G(z, w) = -\delta(z - w), \quad z, w \in \Omega, \\ &G(z, w) = 0, \quad z \in \partial\Omega. \end{aligned}$$

(a) Prove that the solution $u(z)$ can be expressed as:

$$u(z) = \int_{\partial\Omega} f(w) \frac{\partial G}{\partial n_w}(z, w) \, |dw|,$$

where $\frac{\partial G}{\partial n_w}$ is the outward normal derivative with respect to w on $\partial\Omega$.

(b) what are the conditions for the existence and uniqueness of $G(z, w)$ in the context of $\Omega \subset \mathbb{C}$.

(3) Harmonic Measure Representation

(a) Define the harmonic measure $\omega(z, E; \Omega)$ for $z \in \Omega$ and $E \subset \partial\Omega$ by:

$$\omega(z, E; \Omega) = \int_E P_\Omega(z, w)\, |dw|,$$

where $P_\Omega(z, w)$ is the Poisson kernel for Ω.

(b) Prove that the solution $u(z)$ can also be expressed as:

$$u(z) = \int_{\partial\Omega} f(w)\, d\omega(z, w; \Omega).$$

(c) Show explicitly how $P_\Omega(z, w)$ generalizes the Poisson kernel for the unit disk $\mathbb{D}$:

$$P_{\mathbb{D}}(z, e^{i\theta}) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}.$$

(4) Riemann Mapping Theorem

(a) Using the Riemann Mapping Theorem, prove that the Dirichlet problem on any simply connected domain $\Omega \subset \mathbb{C}$ with smooth boundary $\partial\Omega$ can be reduced to solving the Dirichlet problem on $\mathbb{D}$. For $f \in C(\partial\mathbb{D})$, explicitly construct the solution using the Poisson integral formula:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, d\theta, \quad z \in \mathbb{D}.$$